

<http://planetphysics.us/encyclopedia/CategoryTheory.html>

1 Theory of Categories

Category theory can be described as the branch of mathematics concerned with the general, abstract and universal properties and applications of the fundamental [concepts](#) of [category](#), functor between [categories](#) and [natural transformations](#) between functors. A *category* can also be defined as a mathematical interpretation of the [theories of abstract category](#), or [ETAC](#). A [topos](#) is often considered as a special [type](#) of category subject to the [topos axioms](#), and thus based upon a (commutative) Heyting logic.

1.1 Introduction: Basic concepts

Category theory has developed, and is now being further developed, very rapidly in comparison with most of the older branches of mathematics, with the notable exception of Topology, certain aspects of Geometry and Number theory which experienced recently most remarkable advances. The official birthdate year of Category Theory is 1945, even though an earlier, published report in 1943 utilized categorical concepts.

First of all, a *category* consists of arrows, called *morphisms* subject to a very small set of basic category theory axioms (which is as small as only four axioms in some recent formulations). Various chains, or geometric forms, composed of such arrows are called (categorical) *diagrams*. From a logical point of view, morphisms may also be considered as [relations](#), thus generalizing the concept of set-theoretical mapping or [function](#). Therefore, any logical [relation theory](#) may also be formalized in terms of morphisms, and any metalogic (metatheory) of relations among relations may be formalized in terms of categories and functors. One can then also say that “*mathematics is about relations between relations*”.

Charles Ehresmann—one of the founders and developers of category theory in Europe— in evaluating the role played by category theory in modern mathematics pointed out that the concepts of morphism (arrow, with the specific examples of mapping and mathematical function) and mathematical *structure* are the key notions of all modern mathematics, whereas that of a set or *object* is relegated to a secondary, less important role. Morphisms of structures such as monoids, semigroups, groups, rings, [modules](#), [vector spaces](#), [groupoids](#), [topological spaces](#), and so on ‘preserve the basic structure’, thus allowing comparisons to be made between different mathematical objects with the ‘same’ structure.

1.2 Examples of morphisms and categories

As well-known examples, one considers morphisms of groups defined as *group homomorphisms*, and morphisms of topological spaces as *homeomorphisms*. Interestingly, morphisms between groupoids are still being called ‘homomorphisms’ even when such objects possess a topological structure as well. Moreover, groupoids are also regarded as a specific type of categories with all invertible morphisms, or natural generalizations of groups as a no-

tion of ‘group with many [identities](#)’, and they play fundamental roles in [algebraic topology](#). By analogy one might then expect also that Barry Mitchell’s concept of a ‘*ring with many objects*’, and also its generalizations to [algebroids](#) may play, respectively, important roles in [algebraic](#) Geometry and Number Theory. There are two important differences between groupoids and groups. One is that, unlike groups, groupoids have a partial multiplication, and the other difference is that the condition for two elements to be composable is a geometric one (namely the end point of a groupoid arrow is the starting point of another arrow). Such a partial multiplication law that one has for groupoid arrows may be thought of as a kind of “group with many identities”, that reduces to the particular case of a group when there is only one identity. The other very important difference between groups and groupoids is that the geometry, or topology, underlying groupoids is that of directed [graphs](#), whereas the geometry underlying groups is simply that of *based sets*, that is sets with a chosen base point, or objects with *discrete topology* (“set dust”). It is thus quite evident that graphs are far more interesting than sets, and can reflect more geometry—or by considering their connectivity properties in the general case—are endowed with a topological structure. One can also think that the objects of a groupoid allow for the addition of a spatial component to the simpler objects of group theory that may be considered to have only algebraic structures. Note, however, that this simplification applies only to groups other than topological ones; thus one expects a closer connection between a topological group and a [topological groupoid](#) because of their endowment with a topological structure consistently added to their algebraic structure. Furthermore, one may expect even closer links, for example, between a [locally compact groupoid](#) equipped with an associated [Haar measure](#) and a locally compact group also equipped with a Haar measure; the latter argument carries over to the relationships between a [quantum groupoid](#) and a [quantum group](#).

Last-but-not-least, the so called ‘trivial’ groupoid structures arise from the presence of equivalence classes that are important in most [fields](#) of mathematics that involve classification or equivalence relations, and are therefore of fundamental importance in mathematics.

Second level arrows—those between categories—are called *functors*, and third level arrows between functors are called *natural transformations*, again subject to specific naturality conditions such as commutativity of diagrams. The third level arrow is the more powerful concept in comparison with either functors or morphisms, and of course, the functor is a more powerful concept than a morphism.

On the first conceptual level, mathematical categories provide a most convenient, universal-conceptual ‘language’ founded on the notions of category, functor, natural transformation and functor category, albeit at such an abstract and universal level that many classical mathematicians chose to dub it—without any strong justification—as “abstract nonsense”. Nevertheless, there are also *categories with structure*, as well as *enriched categories*, [algebraic categories](#), categories of categories, [2-categories](#), [double categories](#), and so on; category theory can be therefore also as a kind of *metatheory* endowed with different structural levels that are all consistent and natural, in the sense of involving commutativity. Upon imposition of additional (*Ab1* to *Ab6*) axioms such commutative super-structures become Abelian categories that generalize or extend the universal properties of categories of Abelian, or com-

mutative, groups; in fact, half of such axioms are obtained by merely ‘inverting the arrows’—which is called (categorical) *duality*. Whereas the last 50 years have been dominated by developments in (commutative) [homology theory](#) (or Homological Algebra) and Abelian category theory, there is currently occurring a very rapid development of [Non-Abelian Algebraic Topology](#), ([NAAT](#)) in modern mathematics. Such developments have close connections to recent results in [non-Abelian theories](#) in [mathematical physics](#), and also have potential impact on Topological [quantum field theories](#) ([TQFT](#)), [HQFTs](#) and physical mathematics.

However, on a different level, when considered as a further sophistication of algebraic topology, category theory embraces not only algebraic and topological structures, but also geometric and analytic ones. At still higher levels, category theory provides natural means to define higher dimensional structures such as [higher dimensional algebra](#) ([HDA](#)) that open completely new avenues of mathematical research of recent, substantial interest in mathematical physics, and especially in the development of quantum gravitation and/or [superstring](#) theories.

On the other hand, at a fundamental level, the universal categorical concepts of [adjoint functors](#) and [adjointness](#) semantics may provide a unique foundation of mathematics and algebraic theories, well beyond set theory with its known limitations and problems.

1.3 Category theory applications

Among the important category theory applications in mathematics itself are:

- Applications in the Foundations of Mathematics
- Algebraic Topology applications
- Algebraic Geometry applications
- Applications to Number theory
- Applications to rings and modules
- Applications to the theory of Abelian groups
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Other category theory applications are in:

- Theoretical and Mathematical Physics
- Mathematical Biophysics, [complex systems biophysics](#) and Theoretical Biology
- Theory of Logic Algebras
- Categorical logic
- Algebraic logic
- [computer](#) theory
- [quantum operator algebra](#) (QOA)
- Theoretical Ecology and Environmental sciences
- Genomics and Interactomics

1.4 An Index of categories

A partial list of various types of categories.

1.5 A Category theory index

A partial [index of the theory of categories](#).

1.6 An Index of Algebraic Geometry

1.7 An Index of Algebraic Topology

1.8 Bibliography of category theory and its applications

An extensive, but not complete, literature on category theory.

More to come...